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Landau damping: half a century with the great discovery

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Abstract. A brief summary is given of the early studies of Landau damping, followed by a discussion of the issues of singularities in the distribution function, reversibility and nonlinear constraints. A difference is emphasized between the evolution of a single-scale localized Langmuir perturbation and a long quasimonochromatic wavetrain. Applicability conditions of the quasilinear approximation are discussed. Examples of the use of the concept of Landau damping in hydrodynamics, astrophysics and other systems are presented.

1. Introduction

Landau damping can be defined as damping of a collective mode of oscillations in a plasma where collisions between the charged particles are negligibly rare. This phenomenon was predicted in 1946 [1] for Langmuir oscillations [2]. Since then, its presence has been identified in essentially all other modes of collective oscillations in plasma. Various modifications and refinements associated with non-Maxwellian particle distributions, background plasma non-uniformities, magnetic fields, multiple plasma species, nonlinear effects, etc have been made; Landau damping is a concept permeating the whole fabric of modern plasma physics.

The definition of Landau damping given here contains a non-trivial concept: the concept of separability of 'collective modes' and particle collisions. The subtlety of the issue is rooted in that, in a medium where particles are interacting via Coulomb forces, it is not so easy to distinguish between the binary particle collisions and interaction of particles with plasma waves (collective modes). When Landau, in his earlier work [3], derived a collision integral for a fully ionized plasma, he had to truncate divergences in this integral at large impact parameter by arguing that Debye screening of a Coulomb field occurs there. The very reference to Debye screening implies that collisions are, in fact, affected by many-body effects, and are not just binary 'hard-ball' collisions. The issue of a uniform treatment of both the particle collisions and collective modes is still one under discussion (see, e.g. a recent survey by Klimontovich [4] and references therein). We will, however, concentrate on situations where collisions are decisively insignificant and the plasma can be adequately described by the Vlasov equation [5], with only 'smoothed' self-consistent electromagnetic fields taken into account. This is exactly the situation considered by Landau in his original paper of 1946.

During the first 10–15 years after Landau's discovery, his paper was cited and used in only a few publications because of the absence of research programmes in hot collisionless plasmas. Among these early publications was the paper by Bohm and Gross [6] where the electron distribution function was represented as a superposition of monochromatic beamlets. An important observation made in this paper was that, for a non-Maxwellian distribution, an instability may occur. A similar conclusion was drawn by Akhiezer and Fainberg [7]. Van

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Kampen [8] showed that the solution of the initial value problem can be represented as a superposition of a continuous set of singular eigenfunctions. Bernstein, Greene and Kruskal [9] constructed exact nonlinear solutions (so-called BGK modes) in which Landau damping is absent. To the author's knowledge, this was the first publication where the term 'Landau damping' was used.

Generally, in the early years, a lot of attention was paid to interpretation of the singularities that appear in some versions of the theory, in particular, in the analysis of perturbations of the distribution function by a monochromatic wave. In fact, these 'singularities' are, to a great extent, fictitious, not stemming from the physics of the initial-value problem but rather appearing in specific types of its mathematical description (see sections 2 and 5 of this paper).

A nice intuitive interpretation of Landau damping and its nonlinear limits was presented by Dawson in 1961 [10]. First dedicated experiments were carried out by Malmberg and Wharton and have clearly demonstrated the reality of Landau damping [11].

An explosion of interest in Landau damping occurred in the late 1950s–early 1960s, when large-scale fusion research began in several countries and it was realized that Landau damping may strongly affect the phenomenon of anomalous plasma losses from fusion devices, be responsible for the formation of high-energy tails of particle distribution functions, cause fast relaxation of charged particle beams, etc. This realization propagated very quickly to the community of space physicists. Nowadays, approximately every third paper on plasma physics and its applications contains a direct reference to Landau damping (although citations of the original paper [1] are—quite naturally for a broadly recognized effect—rather rare). The concept of Landau damping is widely used in the studies of ensembles of gravitating objects, in the mechanics of continuous media, in elementary particle physics and many other areas of science. Its description and discussion can be found in a number of textbooks, including one of the volumes of the 'Course of Theoretical Physics' by Landau and Lifshitz [12].

In a short paper, it is impossible to mention all the important features of this phenomenon, let alone its innumerable specific applications. We limit ourselves to the following. In sections 2–4, we discuss an interesting and relatively little known problem of the evolution of a single-scale initial perturbation; we show that there are not any intrinsic singularities in this problem; we also touch upon the issues of reversibility and nonlinear limitations. In section 5, we consider more canonical aspects of Landau damping related to the behaviour of separate Fourier harmonics. In section 6, the status of quasilinear theory is briefly summarized. Sections 7 and 8 are devoted to use of the methodology, developed in the context of Landau damping, in other areas of science. A summary is given in section 9.

2. Damping of a localized Langmuir perturbation: regularity versus singularity

We have already briefly touched upon the issue of singularities associated with Landau damping. In this section, we consider the initial value problem for a localized Langmuir perturbation (i.e. the problem treated in the original Landau paper). We assume that the electron plasma is initially 'stirred' in a volume with a characteristic size L (i.e. in the spatial Fourier decomposition of the initial perturbation of the distribution function $k \sim 1/L$). Considering only small perturbations and linearizing the Vlasov equation, one finds

$$\frac{\partial \delta f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \delta f}{\partial \boldsymbol{r}} + \frac{e}{m} \frac{\partial \varphi}{\partial \boldsymbol{r}} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0 \qquad (1)$$

$$\nabla^2 \varphi = 4\pi e \int \delta f \, \mathrm{d}^3 \boldsymbol{v}. \qquad (2)$$

Initial conditions are defined by the initial perturbation of the distribution function, $\delta f|_{t=0} \equiv \delta f_0(\boldsymbol{r}, v)$. We assume that δf_0 is an analytic function of its arguments. The same assumption is made with respect to $f_0(\boldsymbol{v})$.

When the system is allowed to evolve according to equations (1) and (2), there is no point where singularities can appear: the coefficients in the linear first-order equation (1) are analytic functions, as is the initial perturbation $\delta f_0(\boldsymbol{r}, v)$. In other words, singularities are by no means a property of a plasma but appear in some specific mathematical treatments of the problem.

It is instructive to qualitatively describe the evolution of an initial perturbation with a scale-length L considerably exceeding the Debye radius r_D. We avoid the Fourier transform and will describe the evolution of the perturbation in a real space. To obtain a Langmuir mode of a cold plasma, one takes the first two moments of equation (1), neglecting the thermal spread. In this fashion one finds a Langmuir perturbation of the form

$$\varphi = \Phi(\boldsymbol{r}) \cos(\omega_\mathrm{p} t + \Psi(\boldsymbol{r})) \tag{3}$$

with the functions Φ and Ψ depending on the initial conditions and varying at the scale L. In this approximation, the perturbation does not damp.

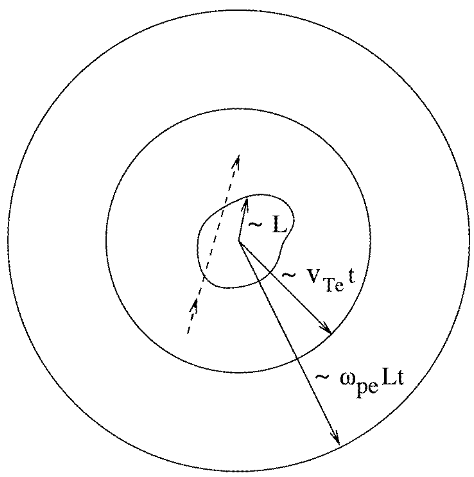

Figure 1. Spatial structure of the perturbation at $t > L/v_\mathrm{Te}$: the centremost zone of the size $\sim L$ is occupied by Langmuir oscillations; the next zone of the size $\sim v_\mathrm{Te} t$ is occupied by the initially perturbed thermal electrons (i.e. the electrons which initially were situated within the size L); the zone of the size $\sim \omega_\mathrm{pe} L t$ is occupied by resonant electrons that crossed the perturbation and carried away some energy. The broken line shows a trajectory of one of the thermal electrons crossing the zone of the perturbation; after they leave the perturbation, their motion differs by only exponentially small corrections from the unperturbed motion.

The structure of the mode (3) is determined by the initial values of the two moments of the perturbation of the electron distribution function, the density and the average velocity. Generally speaking, an initial perturbation of the distribution function cannot be reduced to these moments; this additional part of the initial perturbation is transported by the thermal electrons away from the initial volume and gradually spreads over a much bigger volume. At the time $t \gg L/v_\mathrm{Te}$, it occupies a volume of size $\sim v_\mathrm{Te} t$. This is a slowly varying perturbation, with a spatial scale much greater than the Debye radius (figure 1). It is therefore, quasineutral: the quasineutrality is provided by adjustment of the density of the local thermal electrons by the ambipolar potential that is formed to maintain quasineutrality; in other words, there will be a potential perturbation that expands with electron thermal velocity.

The oscillating field (3) will be continuously traversed by thermal electrons. Consider the motion of one such electron. It approaches the oscillating potential zone with an unperturbed velocity. When inside the zone of oscillation, it acquires oscillatory motion and contributes to the density perturbation that sustains the Langmuir mode. However, as the thermal velocity is much smaller than $L\omega_{\mathrm{p}}$, it leaves the perturbation with the velocity essentially the same as it had when approaching the perturbation. This is because the electric field acting on it is rapidly oscillating, with only a slow variation of amplitude and phase along the particle trajectory. Accordingly, the net result, when the thermal electron leaves the perturbation, is exponentially small.

The behaviour is different for the particles with

$$v \sim L\omega_{\mathrm{p}} \tag{4}$$

which cross the oscillation within a time comparable to, or shorter than, the Langmuir period. For them, the net gain (loss) of energy is substantial. These particles are the ones responsible for Landau damping: on average, they carry away some energy. These energy-absorbing particles, after having crossed the zone of the Langmuir wave, propagate with a large velocity away from the wave region. At time t, they fill a quasispherical volume with radius $\sim vt \sim L(\omega_{\mathrm{p}}t) \gg L$. At any particular point $r \gg L$, they occupy only a very small solid angle, o, in velocity space (an angle at which the perturbation region is seen from this point)

$$o \sim \left(\frac{L}{r}\right)^2 \ll 1. \tag{5}$$

Under the action of these particles, the Langmuir oscillation amplitude gradually decreases by Landau damping. Fast ‘resonant’ particles also create a quasistatic potential perturbation, but it is small because the density of fast particles is small.

In our discussion, we assume that there is a clear separation of time scales: the Langmuir period is much smaller than the transit time of a thermal electron over the scale L, which in turn is much less than the damping time

$$\frac{1}{\omega_{\mathrm{pe}}} \ll \frac{L}{v_{\mathrm{Te}}} \ll \frac{1}{\gamma}. \tag{6}$$

For a purely Maxwellian electron distribution, the damping rate is exponentially small and is of little interest. However, in a number of situations, high-energy tails of the distribution function are present and the damping rate becomes larger. We will assume that the damping rate is a free parameter constrained by equation (6).

According to figure 1, at time $\sim 1/\gamma$, there is a well localized oscillating Langmuir perturbation occupying the zone $\sim L$ near the origin; there is a zone with a size v_{Te}/γ where initial perturbations of the distribution function of thermal electrons are present; and there is a much larger zone $r \sim v/\gamma \sim L(\omega_{\mathrm{p}}/\gamma)$ where the fast particles perturbed by the oscillating field have propagated. This real-space description (not using the Fourier transform) has also proved useful in studies of localized ion-acoustic perturbations [13].

3. Reversibility versus irreversibility

The problem we considered in section 2 shows that the energy initially localized in the Langmuir oscillation in a limited spatial volume gets gradually transferred to fast electrons, with velocities of order of the characteristic phase velocity; these fast electrons smear the energy over a huge space, much greater than the zone occupied by the Langmuir mode. This process with respect to the initial Langmuir mode, can certainly be called ‘damping.’ Still, formally speaking, the system of equations (1) and (2) possesses the property of reversibility and, if inversion of time is made at a certain point, the system will return to the initial state. One should, however, remember that, according to equation (5), the area of velocity space where the perturbation of the distribution function is different from zero, is very small, so that even minor effects may give rise to a loss of the phase memory. Another important observation is that the reversal of particle motion should be performed in a very large volume with the size $r \gg L$. Accordingly, any cause that produces sufficiently strong perturbations with a correlation length smaller than r, will make the system irreversible. This is a very weak requirement. If, for instance, some large-scale electric fields with a correlation length only a few times less than r are present, their amplitude is to be such as to cause a deflection of the electrons from the straight trajectory by an angle $\sim o^{1/2}$; in other words, the field amplitude should be greater than only $mv^2 o^{1/2}/er < mv\gamma o^{1/2}/e \sim mv\gamma L/re$ (see equation (5)) and the energy density, accordingly, be greater than only $mnv^2(L/r)^4$, with $r/L > \omega_{\rm p}/\gamma$. Of course, Coulomb collisions can also make the process irreversible. This happens when they scatter fast particles by the angle $o^{1/2}$.

4. Linearity versus nonlinearity

In the system that we have discussed in section 2, nonlinear effects are relatively insignificant: the energy exchange between the particles with $v \sim L\omega_{\rm p}$ and the oscillating electric field begins to deviate from the predictions of a linear theory only if the velocity perturbation becomes of the order of the velocity itself, i.e. for

$$\frac{eE}{m\omega_{\rm p}} \sim \omega_{\rm p} L. \tag{7}$$

This requires electric fields so high that, in fact, the nonlinearity associated with hydrodynamic motion of a cold plasma begins at a comparable E. In this respect, it is not surprising that there was no discussion of nonlinear effects in the original paper by Landau: their relative insignificance in the initial value problem was clear from the outset.

This conclusion is in stark contrast with the 'standard' picture of the damping of a single spatial harmonic, where nonlinearity turns on relatively early. However, this is merely a reflection of the fact that nonlinearity is determined by the whole set of Fourier harmonics, not by a single harmonic. We summarize the corresponding differences in the next section.

5. Perturbations in the form of a single spatial harmonic

The previous three sections dealt with an assessment of Landau damping based on the equations written in spacetime variables. One particular observation made was that there are no singularities in this initial-value problem. This approach certainly has a number of merits. On the other hand, an approach based on the spatial Fourier decomposition of the initial perturbation, the most traditional one, used in particular by Landau himself, sheds light on many other aspects of the problem, as we now briefly discuss.

For a spatial Fourier harmonic of the form $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$, the set of equations (1) and (2) yields

$$\frac{\partial \delta f_{\boldsymbol{k}}}{\partial t} + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}\delta f_{\boldsymbol{k}} + \mathrm{i}\frac{e\varphi_{\boldsymbol{k}}}{m}\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = 0 \tag{8}$$

$$-k^2\varphi_{\boldsymbol{k}} = 4\pi e\int \delta f\,\mathrm{d}^3\boldsymbol{v}. \tag{9}$$

This is also a linear evolutionary set of equations (the first time-derivative of the distribution function at some instant of time is expressed via this function at this instant of time) and the initial value problem for this set does not have any singularities (if the unperturbed distribution function, as well as the initial perturbation, are regular). As was pointed out by Landau, a possible way of solving the initial value problem is to use the Laplace transform. In this way, by making inverse Laplace and Fourier transforms, one finds a solution of a Cauchy problem for the set (1) and (2).

The analysis of the evolution of a 'pure' spatial harmonic has two sides. On the one hand, it can be used as just mentioned in the formal sense to study the evolution of the initial perturbation, with the understanding that, in such a problem, an artificial separation of one Fourier harmonic for the purpose of discussing singularities, nonlinearities, etc is not warranted by the physical nature of the problem. All these issues can be properly addressed only for a complete perturbation in real space and time. On the other hand, one can meet problems where a real perturbation has the form of a quasimonochromatic wave and then, of course, the analysis of the properties of a wavepacket becomes a physics task of its own. Taking now this second approach, we consider first an evolution of an initial state that was 'prepared' as an infinitely long purely sinusoidal ($\propto \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$) perturbation (with a clear understanding that such perturbations do not exist in a real world).

For $kr_{\mathrm{D}} \ll 1$, after an initial transitional period with a duration $\sim 1/kv_{\mathrm{Te}}$ (an analogue of the time $\sim L/v_{\mathrm{Te}}$ in the problem of a single-scale perturbation of section 2), a Langmuir wave is 'filtered out' and begins its slow decay. Eventually, the coherent part of the perturbation decays and the initial electrostatic energy gets concentrated in the perturbation of the kinetic energy of electrons with velocities close to $v_{\|} = \omega_{\mathrm{p}}/k$, sometimes called the 'resonant point.' The velocity range is

$$\Delta v_{\|} \sim v\left(\frac{\gamma}{\omega_{\mathrm{p}}}\right) \tag{10}$$

where the symbol '$\|$' refers to a direction along the wavevector. As γ is small, this interval is narrow and the perturbation of the distribution function in it contains a large parameter $\omega_{\mathrm{p}}/\gamma$. Therefore, if one deals with one spatial harmonic whose damping rate is small, one finds a rather restrictive linearity condition

$$\frac{eE}{m\omega_{\mathrm{p}}} < \left(\frac{\omega_{\mathrm{p}}}{k}\right)\left(\frac{\gamma}{\omega_{\mathrm{p}}}\right)^2 \tag{11}$$

(cf (equation (7)). However, as we have already emphasized, its restrictiveness is due to the fact that we consider only one spatial Fourier harmonic; for a broad spectrum, the condition is much less restrictive.

Another way of looking at condition (11) is that the damping should occur before the electron (moving with velocity ω_{p}/k) gets displaced by a distance $\sim 1/k$ with respect to the potential profile of a Langmuir wave. The characteristic time for such a displacement (a bounce time) is

$$\tau_{\mathrm{b}} \sim \sqrt{\frac{m}{eEk}}. \tag{12}$$

If condition (11) breaks down, the phase mixing of resonant electrons causes flattening of the distribution function near the point $v_{\|} = \omega_{\mathrm{p}}/k$ within the time shorter than $1/\gamma$ and the damping vanishes. This is a process described by Mazitov and O'Neil [14, 15]. The final state is a kind of a BGK mode [9].

This situation with a singularity in the linear theory comes to its extreme if one looks for the solutions that are harmonic not only in space but also in time, $\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$, with ω real. For such a solution, the distribution function becomes singular at a resonant point

$$\omega - kv_{\|} = 0. \tag{13}$$

In early analyses of the physical mechanism leading to Landau damping the presence of this apparent singularity was a source of confusion (note that there is not such singularity in Landau's analysis!). Sometimes, attempts were made to remove the singularity by introducing collisional friction into electron equations of motion (see, e.g., [6]). This procedure, indeed, allowed one to remove the singularity and even to obtain a correct expression for the damping rate, independent of a collision time τ. One should, however, remember that, in a plasma with rare collisions, a collisional model predicts a very different effect of the wave on the resonant particles than the collisionless (Landau) model: in the collisionless model, the distribution function gets distorted in the range (10) whereas in the collisional model the distribution function gets distorted in the range $\Delta v_\parallel \sim v/\omega_\mathrm{p}\tau$, which is much narrower (for weak collisions). This is not just a cosmetic difference: it leads to very different predictions with regard to the onset of nonlinearity.

One can note in passing that, for the Coulomb collisions described by the Landau collision operator, the collision time scales as $\tau \sim \tau_0(\Delta v_\parallel/v)^2$, where τ_0 is a collision time for 90° scattering. Of course, if collisions are frequent, $\tau < 1/\gamma$, they become important in the analysis of the damping mechanism.

For the purpose of a formal decomposition of an arbitrary initial perturbation of a distribution function over a complete set of eigenfunctions proportional to $\exp(-\mathrm{i}\omega t)$, with ω real, one can use a powerful method suggested by Van Kampen [8] within which one deals with a continuous spectrum of singular eigenfunctions forming a complete set

$$g(v_\parallel) = \frac{8\pi e^2}{mk} f_0(v_\parallel)\left(\frac{p}{\omega - kv_\parallel} + \lambda\delta(\omega - kv_\parallel)\right) \tag{14}$$

with λ being a free parameter (at given ω and $v_\parallel$). One should, however, be cautioned against a direct use of one eigenfunction (14) as a real physical entity: the presence of a singularity makes this eigenfunction very sensitive to even infinitesimal external perturbations of various kinds, e.g. to very weak Coulomb collisions, which smooth eigenfunctions over the finite velocity domain and cause their mix-up. So, Van Kampen modes are objects that can be effectively used only within a convolution scheme (that includes integration over ω).

With all these comments made, consider now a more realistic object: a quasimonochromatic wavepacket propagating in a plasma (figure 2), with $kL \gg 1$. The fact that it has a finite extent in space gives rise to the appearance of a new parameter of the dimension of time in the problem, the transit time L/v of resonant particles through the packet. This time should be compared with the time of linear damping, $1/\gamma$ and the bounce time τ_b (12) of a resonant particle. There are the following three possible situations [16]: (1) if the bounce time is long compared to the linear damping time, $1/\gamma < \tau_\mathrm{b}$, a packet of any length damps according to the predictions of the linear theory; (2) if the bounce time is short compared to the damping time but long compared to the transit time, $L/v < \tau_\mathrm{b} < 1/\gamma$, the linear theory still works, because the time during which the particles are exposed to the action of the wavepacket is too short to cause any distortions of the distribution function; (3) if the bounce time is short compared to the transit time, $\tau_\mathrm{b} < L/v$, one enters the domain where the aforementioned nonlinear saturation of damping occurs. In this regime the packet does not damp uniformly over its length, but is rather being gradually 'eaten up' from the edge through which resonant particles enter the packet [16]. A similar problem was studied for the whistler mode [17].

Another situation where almost periodic perturbations are important players, is that of the echo [18, 19]. This phenomenon allows one to catch information 'hidden' in a strongly 'tangled' distribution function formed after the damping of the initial wave and make it again visible by imposing another harmonic perturbation, further in the direction of motion of the affected particles. A nice discussion of this phenomenon can be found in Kadomtsev's book [20].

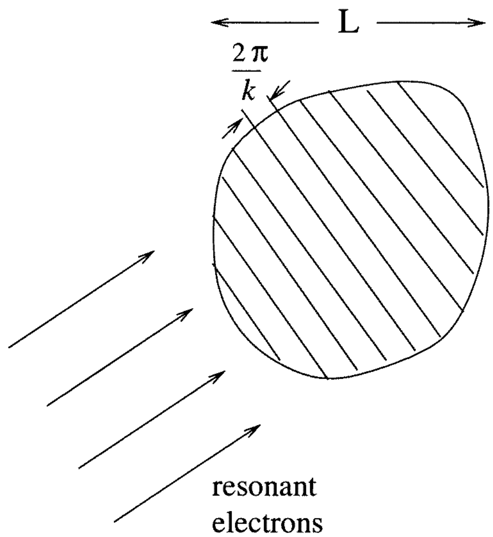

Figure 2. A wavepacket traversed by resonant electrons. If the condition $\tau_b < L/v$ is satisfied, the wavepacket is eroded from the left (instead of experiencing a uniform decrease of amplitude).

6. Landau damping in weak turbulence theory

The first step in incorporating the effects of Landau damping into weak turbulence theory was made in papers on 'quasilinear theory' [21, 22]. This theory takes into account the inverse effect of plasma waves on the particle distribution function, i.e. it makes one step forward beyond the purely linear approximation. In the concept of homogeneous steady state turbulence (i.e. turbulence occupying a volume of many wavelengths and varying at a time scale long compared to the wave period), representation of perturbations in terms of a superposition of harmonic waves proves most natural. Here, issues of the physical interpretation of the evolution of quasimonochromatic waves become important. An assumption made in the derivation of quasilinear theory is that the spectrum is sufficiently broad, so that no trapping of particles would occur near the potential minimum of a particular component of the spectrum. This condition reads

$$\sqrt{\frac{eE}{km}} \ll v\left(\frac{\Delta k}{k}\right). \tag{15}$$

This condition also means that the width of the frequency spectrum seen from the frame moving with a velocity corresponding to the centre of the resonant interval, ω_p/k, is large compared to the inverse time of a quasilinear evolution of the distribution function. In this case, a random-phase approximation becomes applicable and one can write down equations in terms of the spectral energy density of the waves $W_{\boldsymbol{k}}$. For Langmuir waves, the corresponding equations read

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial f}{\partial v_\beta} \qquad D_{\alpha\beta} = \frac{8\pi^2 e^2}{m^2} \int \frac{k_\alpha k_\beta}{k^2} W_{\boldsymbol{k}} \delta(\omega_p - \boldsymbol{k}\cdot\boldsymbol{v})\, \mathrm{d}^3\boldsymbol{k}$$

$$\frac{1}{W_{\boldsymbol{k}}}\frac{\partial W_{\boldsymbol{k}}}{\partial t} = \frac{4\pi^2 e^2}{mk^2} \int \boldsymbol{k}\cdot\frac{\partial f}{\partial \boldsymbol{v}} \delta(\omega_p - \boldsymbol{k}\cdot\boldsymbol{v})\, \mathrm{d}^3\boldsymbol{v}. \tag{16}$$

Using the concept of broad spectra allows one to make further steps in developing the theory of a weakly turbulent plasma, by taking into account higher-order nonlinearities. One of the processes that appears in the next (after quasilinear) order of the perturbation theory is so-called nonlinear Landau damping. It corresponds to the condition where the particles experience resonance with a beat wave, i.e. the following condition is satisfied

$$\omega(\boldsymbol{k}_1) - \omega(\boldsymbol{k}_2) = (\boldsymbol{k}_1 - \boldsymbol{k}_2) \cdot \boldsymbol{v}. \tag{17}$$

This process may be important, for example, if the linear Landau resonance corresponds to waves with a superluminous phase velocity, whereas the beat wave has velocity comparable with the thermal velocity.

The applicability of quasilinear theory as described by equations (16) is limited, on the one hand, by the necessity to have a broad spectrum of the waves and, on the other hand, by the constraint that the higher-order corrections are insignificant. For discussion of these and other constraints see [23] and references therein.

There are innumerable examples of important and interesting applications of quasilinear theory. We mention two of them here. The first is collisionless relaxation of a relativistic electron beam in a plasma. For a strongly relativistic electron beam, the condition of an energy exchange with a wave characterized by the wavevector $\boldsymbol{k}$ is: $\omega - c\boldsymbol{k} \cdot \boldsymbol{p}/p = 0$ where $\boldsymbol{p}$ is the momentum of a relativistic electron. Note that only the direction of the electron momentum is defined by this condition, not its absolute value. If a Langmuir wave propagating along the beam becomes unstable, it necessarily causes not only diffusion of one part of the beam electrons towards lower energies but also diffusion of the other part to higher energy. This circumstance was demonstrated experimentally in an almost textbook fashion by Arzhannikov *et al* [24].

The second is so-called alpha channeling [25]. It is well known that 3.5 MeV alpha particles formed in a fusion reactor deliver most of their energy to electrons, which, in turn, slowly transfer it to plasma ions. It is conceivable that there exist waves that will be driven by the alpha particles and absorbed directly (via Landau damping), by the plasma ions, without binary collisions being involved in this process. This would considerably improve performance of a tokamak reactor. There exist experimental indications that such a scenario is indeed possible [25], especially if waves from the external sources are used to affect the evolution of the distribution function. One can call this approach 'phase-space engineering.'

7. Analogues of Landau damping in hydrodynamics

The methodology developed for the analysis of Landau damping of a single spatial harmonic has proven successful in the studies of sheared hydrodynamic flows and magnetohydrodynamic (MHD) waves. A typical problem is that of the propagation of Alfvén waves in a slab where the unperturbed magnetic field has y and z components varying in the x direction. For a wave of the form, $\exp(-\mathrm{i}\omega t + \mathrm{i}ky)$, the equation for the x component of the velocity perturbation reads (see, e.g. [26])

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((\omega^2 - \Omega^2)\frac{\mathrm{d}U}{\mathrm{d}x}\right) - k^2(\omega^2 - \Omega^2)U = 0 \tag{18}$$

where $\Omega^2(x) = k^2 B_y^2(x)/4\pi\rho(x)$. The singularity in this equation has the same origin as the one that appears in the problem of Landau damping for solutions with a purely harmonic time dependence (section 5). A reliable way of assessing this problem is to consider an initial value problem, and then use a Laplace transform technique [27]. A close similarity with Landau damping then becomes obvious. For some profiles of the magnetic field, in particular ones where Ω is almost constant inside some radius and then drops sharply to zero, weakly damped oscillations are possible. There, asymptotically, an almost sinusoidal mode $\sim\exp(-\mathrm{i}\omega t)$

indeed emerges from the initial perturbation. The damping mechanism consists in gradual excitation of a narrow band of a continuous spectrum near a ‘resonant’ point, $\omega = \Omega$ (general problems of ‘Alfvén continua’ are nicely discussed in [28]). For smooth radial profiles of the Alfvén velocity, initial perturbations damp very quickly [29] and, generally speaking, in a non-exponential way [27]. Very similar problems exist for other modes in non-uniform plasmas (see, e.g. [30, 31]).

Returning to the ‘resonant’ side of the phenomenon, one can mention that sheared flow may drive instabilities by the mechanism of ‘inverse’ Landau damping. In particular, this mechanism can be responsible for excitation of wind waves on the surface of the water. If in the sheared flow of the wind over the water surface there exists a point where the wind velocity coincides with the phase velocity of surface waves, a Landau-type interaction is possible. Usually, there is not a single point but a broad range of velocities where resonance is present. To drive the instability, the dependence of wind velocity over the height $v(z)$ should be such that $vv'' < 0$, as is usually the case. This mechanism was discovered in the 1950s [32]; its interpretation in terms of Landau damping was given only recently [33, 34].

Analogues of Landau damping exist in multiphase media [35]. In a bubbly fluid with a broad distribution of the bubbles over their sizes, a long-wave acoustic wave can always find bubbles whose radial eigenfrequencies are equal to the wave frequency. The initial coherent acoustic wave then damps, converting its energy into the energy of oscillations of the bubbles near the resonant frequency. This occurs within a time much shorter than the time of the dissipative processes. In a plasma of a solar atmosphere, with randomly distributed magnetic ropes, long-wavelength acoustic waves resonantly excite kink and sausage modes of the ropes and cease to exist as coherent oscillations.

8. Analogues of Landau damping in other systems

During the decades that passed after Landau’s discovery, analogues of Landau damping (or ‘inverse’ Landau damping) have been identified in a number of areas of physics. The most natural generalization was to the physics of gravitating systems: since the gravitational interaction is of the same type as a Coulomb interaction, a collisionless Vlasov equation can be applied to the analysis of equilibria and stability of stellar systems (e.g. [36–38]). One can expect that kinetic phenomena of Landau damping type will be present in such systems. The analysis in this case is complicated by the fact that gravitational forces are universally attractive, so that initial equilibrium states are necessarily non-uniform. Still, many similarities with an electron–ion plasma remain; in particular, bump-on-tail (beam) instability is possible [39].

More recently, the concept of Landau damping made inroads into the field of high-energy physics: in paper [40], effects of Landau damping were invoked to obtain finite production rates of hard photons from a quark–gluon plasma. Landau damping occurs on the exchange quark.

Analogues of Landau damping are possibly present in biological systems. In [41], a system of oscillators coupled via their phases has been considered. The corresponding equation has the form

$$\dot{\vartheta}_{\mathrm{i}} = \omega_{\mathrm{i}} + \frac{K}{N}\sum_{j=1}^{N}\sin(\vartheta_j - \vartheta_{\mathrm{i}}) \qquad (19)$$

where ϑ_{i} represent the phases of the oscillators, ω_{i} their frequencies and K is a coupling constant. For a very large number of oscillators $N \gg 1$, one can switch from a discrete model to a continuous model by introducing the distribution function of the oscillators over the phase angles and eigenfrequencies and write down a continuity equation in the phase space. Thereby one arrives at a set of equations similar to the ones met in the problem of Landau damping. The conclusion is that, for a coupling constant below a certain level, the system always relaxes to an incoherent state. According to the authors, this observation may be relevant to such phenomena as synchronous flashing of fireflies and to a synchronous firing of cardiac pacemaker cells.

9. Summary

The concept of collisionless damping, introduced into modern physics by Landau, is a living concept. This phenomenon plays a decisive role in essentially all branches of plasma physics. In addition, the methodology developed in the studies of Landau damping has proven its effectiveness in the analysis of problems in hydrodynamics, astrophysics and high-energy physics.

Conceptually, Landau damping is not a very simple phenomenon and its interpretation in various specific settings is still a challenging problem. It is no surprise that papers devoted to these interpretations, as well as to developing better concepts of teaching this phenomenon, appear regularly even now, 50 years after the discovery (e.g. [42–49]). Persistent interest in this complex and in many respects paradoxical phenomenon promises new breakthroughs, both in theory and applications.

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